Functional Quantum Theory of Scattering Processes. IV.

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Dynamics of quantum field theory can be formulated by functional equations. To develop a complete functional quantum theory the physical information has to be given by functional operations only. The most important physical information of elementary particle physics is the S-matrix. In this paper the functional S-matrix is constructed for the scattering of relativistic dressed particles, i.e. for particles with structural properties. The basic functional equation is assumed to be derived from a nonlinear spinor field equation with noncanonical relativistic Heisenberg quantization. The initial free dressed many particle states are defined, and the scattering functionals are constructed. By the use of irre ducible representations the equivalence of the functional S-matrix with the conventional Hilbert space definition is shown with respect to an appropriate definition of the functional scalar product. Technical details are discussed in the appendices.

The operator equations of quantum field theory can be replaced formally by functional equations of corresponding Schwinger functionals [1–3]. To give this formalism a physical and mathematical meaning it has to be considered as a mapping between physical state spaces and functional state spaces [4, 5]. Thus one has to develop a complete functional quantum theory, where the complete physical information has to be obtained by functional operations only. This has been proposed in a preceding paper [6]. As has been pointed out in [4], the mapping is possible only for global observables, i.e. the stationary quantum numbers and the S-matrix. The functional definition of the stationary quantum numbers has been discussed in [7], while the S-matrix has been constructed only for spin 1/2 Fermion-Fermion scattering so far [8, 9]. In this paper the functional S-matrix construction is discussed in full generality, i.e. for all types of scattering processes between particles with arbitrary angular momentum. To be able to perform functional calculations of interesting high energy phenomena, the nonlinear spinor equation regularized by noncanonical Heisenberg quantization [10] is assumed to be the basic field equation. But the method can be applied to any quantum model, provided it gives finite results and contains particles with higher angular momentum i.e. composite or dressed particles. In the case of undressed particles the S-matrix definition is reduced to that given already in [8].

1. Fundamentals

The connection between nonlinear spinor theory and its functional map has been discussed in [11]. Therefore the fundamentals may be discussed only in functional space. The appropriate state description is provided by the normal functionals [4, 11]. They are defined by a power series expansion

$$|\Phi(j)\rangle := \sum_{n=0}^{\infty} \frac{1}{n!} \varphi_n \underset{\alpha_1 \;\dots\; \alpha_n}{(x_1 \dots x_n)} j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n) \,|\varphi_0\rangle \tag{1.1}$$

where for double indices and coordinates the summation convention is assumed. The power series functionals occuring in (1.1) are defined in [4, 11, 12]. By (1.1) the functional equation

$$S^{\varkappa}(x)\,[\mathrm{d}_{\varkappa}(x) + G^{\alpha}_{\varkappa}(x-x')\, V_{\alpha}^{\beta\gamma\delta}\, \mathrm{d}_{\beta}(x')\, \mathrm{d}_{\gamma}(x')\, \mathrm{d}_{\delta}(x')] \times |\Phi(j)\rangle = 0 \tag{1.2}$$

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has to be satisfied where $S^{\varkappa}(x)$ is a suitable symmetrization operator[11,13,14] and the definition

$$\mathrm{d}_{\varkappa}(x) := \underset{\varkappa\ \alpha}{F}(x\,z)\, j^{\alpha}(z) + \partial_{\varkappa}(x) \tag{1.3}$$

is used. For a review about the functional space see [11]. Additionally the symmetry conditions

$$\begin{gathered}\mathfrak{P}_h\,|\,\Phi(j)\rangle = p_h\,|\,\Phi(j)\rangle; \quad \mathfrak{P}^2\,|\,\Phi(j)\rangle = m^2\,|\,\Phi(j)\rangle,\\ \mathfrak{G}_\mu\,\mathfrak{G}^\mu\,|\,\Phi(j)\rangle = s(s+1)\,|\,\Phi(j)\rangle; \\ \mathfrak{S}_3\,|\,\Phi(j)\rangle = s_3\,|\,\Phi(j)\rangle,\end{gathered} \tag{1.4}$$

defining the maximal set of quantum numbers with respect to the Poincaré group have to be satisfied by state functionals[7,15]. We do not discuss further quantum numbers arising from gauge groups etc. It is assumed, that the set of solutions of (1.2) (1.4) provides a complete set of physical states of the theory. Generally for physical states one has to distinguish between stationary states and scattering states. For the calculation of stationary states a method has been proposed in [11]. The functional state vectors satisfy homogeneous equations, and it is assumed, that the operator kernels of these equations determine the stationary states completely like in the simpler case of Fredholm equations. On the other hand scattering functionals have to be characterized by initial conditions, arising from the ingoing resp. outgoing particle configurations before or after the scattering has been occured. These initial conditions have to be incorporated in the description of scattering functionals. This has been done for Fermion-Fermion scattering in [8,9,11]. Now we discuss the general case.

2. Scattering Functionals

The initial conditions of scattering states are free many particle states of dressed particles. To construct them we consider a one particle state

$$|\,\Phi_{\mathfrak{K}}(j)\rangle := \sum_n \frac{1}{n!}\,\varphi_n(x_1\ldots x_n,\,\mathfrak{K})\, j(x^1)\ldots j(x^n)\,|\,\varphi_0\rangle \tag{2.1}$$

where $\mathfrak{K}$ means the set of quantum numbers defining the particle state completely. This state can be constructed by applying a creation operator

$$\mathfrak{A}^+(\mathfrak{K}) := \sum_n \frac{1}{n!}\,\varphi_n(x_1\ldots x_n,\,\mathfrak{K}))\, j(x^1)\ldots j(x^n) \tag{2.2}$$

on $|\,\varphi_0\rangle$. From (2.2) immediately follows a many particle state

$$|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle := \mathfrak{A}^+(\mathfrak{K}_1)\ldots\mathfrak{A}^+(\mathfrak{K}_n)\,|\,\varphi_0\rangle \tag{2.3}$$

by repeated application of creation operators on $|\,\varphi_0\rangle$. While the one particle states (2.1) are solutions of (1.2) and (1.4), the many particle states (2.3) do not satisfy (1.2) but only (1.4).

Stat.: The many particle states are solutions of (1.4) but in general not of (1.2).

Proof: We consider first a generalisation of formula (3.6) of [7] which reads:

$$\begin{aligned}\mathfrak{P}_h\,\mathfrak{A}^+(\mathfrak{K}_1)\ldots\mathfrak{A}^+(\mathfrak{K}_n)\,|\,\varphi_0\rangle = \sum_{\varrho_1\ldots\varrho_n} \frac{1}{\varrho_1!\ldots\varrho_n!}\left\{\sum_{\nu_1=1}^{\varrho_1} P_h(x^{(1)}_{\nu_1}) + \cdots + \sum_{\nu_n=1}^{\varrho_n} P_h(x^{(n)}_{\nu_n})\right\}\\ \cdot\,\varphi_{\varrho_1}(x^{(1)}_1\ldots x^{(1)}_{\varrho_1},\,\mathfrak{K}_1)\ldots\varphi_{\varrho_n}(x^{(n)}_1\ldots x^{(n)}_{\varrho_n},\,\mathfrak{K}_n)\, j(x^{(1)}_1)\ldots j(x^{(n)}_{\varrho_n})\,|\,\varphi_0\rangle\,.\end{aligned} \tag{2.4}$$

By observing

$$\tilde{\varphi}_n(p_1\ldots p_n,\,\mathfrak{K}) = \delta(p^{\mathfrak{K}} - \sum_{i=1}^n p_i)\,\tilde{\varphi}_n(p_1\ldots p_n) \tag{2.5}$$

we obtain the relations

$$\begin{gathered}\mathfrak{P}_h\,|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle = (\sum_{j=1}^n p_h^{\mathfrak{K}_j})\,|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\quad \mathfrak{P}^2\,|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle = (\sum_{j=1}^n p_h^{\mathfrak{K}_j})\,(\sum_{l=1}^n p^{h\mathfrak{K}_l})\,|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\,,\\ \mathfrak{G}_\mu\,\mathfrak{G}^\mu\,|\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle = S(S+1)\,|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle,\quad \mathfrak{S}_3\,|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle = S_3\,|\,\Phi(j,\,\mathfrak{K}_1\ldots\mathfrak{K}_n)\rangle\end{gathered} \tag{2.6}$$

where S denotes the quantum number of the total angular momentum (relative to the center of mass of the n-particle system) and S_3 that of the third component.

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The first one is clear. The other ones are obtained by the same procedure. Therefore the first statement is verified.

For the second statement, we observe, that any one particle solution does satisfy only Eqs. (1.2) and (1.4) as no further physical conditions are required.

Denoting the operator of Eq. (1.2) by

$$\mathfrak{D}_h(j,d) = j^{\varkappa}(x)\, P_h(x)\, G^{\alpha}_{\varkappa}(x-x')\, V^{\beta\gamma\delta}_{\alpha}\, d_\beta(x')\, d_\gamma(x')\, d_\delta(x') \tag{2.7}$$

for a one particle state $\mathfrak{A}^+(\mathfrak{R})\,|\varphi_0\rangle$ the equation

$$[m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)]\,\mathfrak{A}^+(\mathfrak{R})\,|\varphi_0\rangle = 0 \tag{2.8}$$

has to be fulfilled. If $|\Phi(j,\mathfrak{R}_1\ldots\mathfrak{R}_n)\rangle$ would be an eigensolution too, a necessary condition would be to satisfy (2.8) also. We show that this is not true. We consider the special case of a two particle state $|\Phi(j,\mathfrak{R}_1,\mathfrak{R}_2)\rangle$. Then we have due to (2.8)

$$[m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)]\,\mathfrak{A}^+(\mathfrak{R}_1)\,\mathfrak{A}^+(\mathfrak{R}_2)\,|\varphi_0\rangle = [(m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)),\mathfrak{A}^+(\mathfrak{R}_1)]_-\,\mathfrak{A}^+(\mathfrak{R}_2)\,|\varphi_0\rangle\,. \tag{2.9}$$

A necessary condition for $\mathfrak{A}^+(\mathfrak{R}_1)\,\mathfrak{A}^+(\mathfrak{R}_2)\,|\varphi_0\rangle$ to be an eigensolution is the vanishing of (2.9). As the one particle state $\mathfrak{A}^+(\mathfrak{R}_2)\,|\varphi_0\rangle$ satisfies only Eqs. (1.2), (1.4) this can be achieved only by

$$[(m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)),\mathfrak{A}^+(\mathfrak{R}_1)]_-\rangle \equiv m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)\,.$$

By direct calculation it can be shown that this is not fulfilled, therefore $|\Phi(j,\mathfrak{R}_1,\mathfrak{R}_2)\rangle$ is no eigenstate.

For higher states we have

$$\begin{aligned}&[m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)]\,|\Phi(j,\mathfrak{R}_1\ldots\mathfrak{R}_n)\rangle\\ &\quad= \sum_{l=1}^{n}\mathfrak{A}^+(\mathfrak{R}_1)\ldots\mathfrak{A}^+(\mathfrak{R}_{l-1})\,[\{m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)\},\mathfrak{A}^+(\mathfrak{R}_l)]_-\,\mathfrak{A}^+(\mathfrak{R}_{l+1})\ldots\mathfrak{A}^+(\mathfrak{R}_n)\,|\varphi_0\rangle\,.\end{aligned} \tag{2.10}$$

In this combination at least the term with $l = n-1$ is $\neq 0$ due to the statement about the two particle states. Therefore in general (2.10) is $\neq 0$ and the many particle states (2.3) are no eigensolutions of (1.2). q.e.d.

The physical reason for this property of many particle states is obvious: As in nonlinear field theories the interaction cannot switched out, only stable one particle solutions may exist. Any many particle solution has to be a new stable one particle solution, i.e. a new dressed particle or a scattering state. For a new stable particle the quantum numbers of the constituents cannot be fixed. Therefore (2.3) has to be a scattering state. But also scattering states in general cannot exist with one fixed free particle configuration, therefore (2.3) cannot be a solution of (1.2). As the operators (2.2) create dressed particles, the following holds

Stat.: The algebra generated by the dressed particle creation operators $\mathfrak{A}^+(\mathfrak{R})$ and destruction operators $\mathfrak{A}(\mathfrak{R})$ is not isomorphic to the algebra of the free particle operators $\mathfrak{A}_f^+(\mathfrak{R})$, $\mathfrak{A}_f(\mathfrak{R})$.

Proof: For free particle states the functional creation operators are given for Spin 1/2 fermions by

$$\mathfrak{A}_f^+(\mathfrak{R}) := \varphi_1(x_1)\, j(x^1)\,. \tag{2.11}$$

Also for higher spin particles analogous local operators may be defined. By direct calculation the commutators and anticommutators of the dressed particle algebra and the free particle algebra may be derived. A comparison shows that a mapping is not possible in general.

As can be seen the dressed particle algebra is much more complicated than the free particle algebra. But this is no serious difficulty as the complete dressed particle algebra is not required for practical calculations. Concerning the connection with physical Hilbert space, the following statement is given:

Stat.: The dressed many particle states in functional space can be mapped on dressed many particle states in physical Hilbert space.

No proof of this statement is given, as a proof depends on the cluster properties of vacuum expectation values, which are not explored satisfactory.

To apply the states (2.3) for the construction of scattering states, it is necessary to consider wave packets of (2.3). These are given by linear combinations

$$|\Phi(j,\alpha)\rangle := C_\alpha(\mathfrak{K}_1 \dots \mathfrak{K}_n)\,|\Phi(j,\mathfrak{K}_1 \dots \mathfrak{K}_n)\rangle \quad (2.12)$$

where α denotes the quantum numbers of the packet. By a suitable choice of wave packets the configurations of free dressed particles can be arranged to have an almost vanishing interaction. This is the starting point of S-matrix construction in ordinary Hilbert space. For the scattering functionals then the following theorem is assumed

Stat.: The scattering functionals $|\Phi^{(\pm)}(j)\rangle$ for an initial or final configuration (2.12) of free dressed particles decompose into

$$|\Phi^{(\pm)}(j,\alpha)\rangle = |\chi^{(\pm)}(j,\alpha)\rangle + |\Phi(j,\alpha)\rangle\,. \quad (2.13)$$

Proof: An explicit proof will not be given, as many assumptions are required. We only give a hint. The proof should run like that in [8], if one observes, that the total Hamiltonian in physical Hilbert space can be decomposed into a spectral representation of free dressed many particle states and an additional interaction energy between these states. Then the considerations of [8] can be applied.

Finally we look for a method of explicit construction of scattering states. Substitution of (2.13) into (2.8) gives with (2.10)

$$\begin{aligned}
&[m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)]\,|\chi^{(\pm)}(j,\alpha)\rangle \\
&\quad = \sum_{l=1}^{n} \mathfrak{A}^+(\mathfrak{K}_1)\dots\mathfrak{A}^+(\mathfrak{K}_{l-1}) \qquad (2.14)\\
&\quad\times [\{m^2 - \mathfrak{D}_h(j,d)\,\mathfrak{D}^h(j,d)\}, \mathfrak{A}^+(\mathfrak{K}_l)]_- \\
&\quad\times \mathfrak{A}^+(\mathfrak{K}_{l+1})\dots\mathfrak{A}^+(\mathfrak{K}_n)\,|\varphi_0\rangle\,.
\end{aligned}$$

If one assumes a power series solution of

$$[m^2 - \mathfrak{D}_h\,\mathfrak{D}^h \pm i\,\gamma]^{-1}$$

one obtains from (2.16) by evaluation of the right side

$$\begin{aligned}
&|\chi(j,\alpha)\rangle = |\chi^{(+)}(j,\alpha)\rangle = |\chi^{(-)}(j,\alpha)\rangle \\
&\quad = \sum_{l=0}^{\infty}\left(\frac{1}{m^2}\,\mathfrak{D}_h\,\mathfrak{D}^h\right)^l \Bigg(\sum_{k=1}^{n}\mathfrak{A}^+(\mathfrak{K}_1)\ \dots\mathfrak{A}^+(\mathfrak{K}_{k-1}) \\
&\quad\times[\{m^2 - \mathfrak{D}_h\,\mathfrak{D}^h\}, \mathfrak{A}^+(\mathfrak{K}_k)]_- \qquad (2.15)\\
&\quad\times \mathfrak{A}^+(\mathfrak{K}_{k+1})\dots\mathfrak{A}^+(\mathfrak{K}_n)\Bigg)\,|\varphi_0\rangle\,.
\end{aligned}$$

This solution corresponds to the common Born series. It may be used if no resonance scattering occur. In the case of resonance phenomena the reciprocal has to be studied more thoroughly.

3. S-Matrix Construction

For the construction of the S-matrix a scalar product definition in functional space is required especially for spinor field functionals. The possible functional spaces and the scalar products of their base vectors are discussed in [4,11,12]. The mapping between functional state space and physical state space is produced by the selection of a definite functional space. This is done by the suitable choice of a weighting functional W. In preceding papers a weighting functional $W = \exp\{-j\,G\,j\}$ has been proposed[6,11,8,9]. If G is squareintegrabel in both coordinates no divergencies occur in the map. But as has been pointed out in [6] no general G valid for all theories can be found. Rather a special G has to be defined for any special theory to provide a physical meaningfull map. Therefore it is of interest to look for an universal weithting factor valid for all theories. To achieve this we introduce the following definition

$$\begin{aligned}
W := \sum_{m=1}^{\infty} \varepsilon^m\,|D_m\overset{\beta_1\ \ \dots\ \ \beta_m}{(y_1\dots y_m)}\rangle\,\delta^{\beta_1}_{\gamma_1}\,\delta(y_1 - z_1)& \\
\dots\delta^{\beta_m}_{\gamma_m}\cdot\delta(y_m - z_m)\,\langle D_m\underset{\gamma_1\ \ \dots\ \ \gamma_m}{(z_1\dots z_m)}|& \quad (3.1)
\end{aligned}$$

with the power functionals

$$|D_m\overset{\beta_1\ \ \dots\ \ \beta_m}{(y_1\dots y_m)}\rangle := \frac{1}{m!}\,j^{\beta_1}(y_1)\dots j^{\beta_m}(y_m)\,|\varphi_0\rangle\,. \quad (3.2)$$

Applying now the expressions for power functional scalar products of[4,11,12] the following theorem holds

Stat.: If $|\Phi_a(j)\rangle$ and $|\Phi_b(j)\rangle$ are state functionals of type (1.1) then

$$\lim_{\varepsilon\to 0}\langle\Phi_a(j)|\,W\,|\Phi_b(j)\rangle = \sum_{\nu=0}^{\infty} C_\nu(a,b)\,\delta^\nu(0) \quad (3.3)$$

is valid.

Proof: The proof is given in appendix II.

The power terms of $\delta(0)$ result of the socalled disconnected graphs. For exponential weights these terms are suppressed, i.e. they are finite. For the universal weight (3.1) they give rise to divergencies. Therefore the scalar product of state functionals cannot immediately defined by (3.3). To avoid this difficulty we therefore define the physical functional scalar product by

Def.:

$$(\Phi_a(j)\,|\,\Phi_b(j)) := \lim_{\varepsilon\to 0}\langle\Phi_a(j)|\,W\,|\Phi_b(j)\rangle/_{\delta(0)=0} \quad (3.4)$$

i.e. this scalar product is defined by the zero order term of the expansion (3.3). This definition provides a suitable map between the functional and physical state space. The following holds

Stat.: Stationary functional states $|\Phi_a(j)\rangle$, $|\Phi_b(j)\rangle$ are orthogonal for $a \neq b$.

Proof: Any stationary state is characterized by a maximal set of quantum numbers. If this set is chosen suitably no degeneracy occurs. This is assumed for the following. Therefore if $a \neq b$, among the set of quantum numbers $a := (a_1 \dots a_n)$, $b := (b_1 \dots b_n)$ at least one pair a_α, b_α is different. As the quantum numbers are defined by the eigenvalues of group generators of the corresponding symmetry group we denote the corresponding operator by

$$\mathfrak{G}_\alpha := j_\varkappa(x)\, G_\alpha(x)\, \partial^\varkappa(x)\,. \tag{3.5}$$

Then we have

$$\langle\Phi_b(j)|\, W\, \mathfrak{G}_\alpha\, |\Phi_a(j)\rangle = a_\alpha \langle\Phi_b(j)|\, W\, |\Phi_a(j)\rangle,$$
$$\langle\Phi_a(j)|\, W\, \mathfrak{G}_\alpha\, |\Phi_b(j)\rangle = b_\alpha \langle\Phi_a(j)|\, W\, |\Phi_b(j)\rangle. \tag{3.6}$$

By direct calculation follows

$$\langle\Phi_a(j)|\, W\, |\Phi_b(j)\rangle = \langle\Phi_b(j)|\, W\, |\Phi_a(j)\rangle^\times \tag{3.7}$$

where $\times$ means complex conjugation. Therefore from (3.6) follows

$$\langle\Phi_b(j)|\, W\, \mathfrak{G}_\alpha\, |\Phi_a(j)\rangle - \langle\Phi_a(j)|\, W\, \mathfrak{G}_\alpha\, |\Phi_b(j)\rangle^\times = (a_\alpha - b_\alpha)\, \langle\Phi_b(j)|\, W\, |\Phi_a(j)\rangle\,. \tag{3.8}$$

As $\mathfrak{G}_\alpha$ leads to an observable quantum number, we may assume $G_\alpha(x)$ to be a Hermiteam operator without restriction. Observing this and the definitions of scalar products between power functionals, by direct calculation follows

$$\langle\Phi_b(j)|\, W\, \mathfrak{G}_\alpha\, |\Phi_a(j)\rangle = \langle\Phi_a(j)|\, W\, \mathfrak{G}_\alpha\, |\Phi_b(j)\rangle^\times \tag{3.9}$$

and therefore from (3.8)

$$\langle\Phi_b(j)|\, W\, |\Phi_a(j)\rangle = 0 \quad (a \neq b)\,. \quad \text{q.e.d.} \tag{3.10}$$

Choosing a suitable normalization one therefore has an unitary map between stationary functional states and physical states. This map can be extended to scattering states. We proof the following

Stat.: If $|a\rangle$ and $|b\rangle$ are base vectors of irreduzible representations in physical Hilbert space, then $|\Phi_a(j)\rangle$ and $|\Phi_b(j)\rangle$ are base vectors of the corresponding irreduzible representations in functional space.

Proof: Irreducible representations are defined by the diagonalization of a maximal set of corresponding group operators. This definition does not depend on the representation space. Therefore if these operators are diagonalized in physical state space, they are diagonalized in functional state space, too.

Stat.: If $|a\rangle$ and $|b\rangle$ are orthogonal base vectors of irreducible representations in physical Hilbert space, then also $|\Phi_a(j)\rangle$ and $|\Phi_b(j)\rangle$ are orthogonal with respect to (3.4).

Proof: The proof runs like that for stationary state functionals. The property that the stationary functionals are solutions of (1.2) is not required.

By the preceding statement the scattering functionals can be treated to

Stat.: Between scattering states $|a^{(\pm)}\rangle$ and $|\Phi_a^{(\pm)}(j)\rangle$ an unitary mapping is provided by (3.4).

Proof: Any physical state has to be a base state of a representation of the corresponding symmetry groups. In contrary to stationary states, the scattering states generally do not generate irreducible but reducible representations. As the entire representation space is given by the direct sum of the base vectors of irreducible representations, it follows that any scattering state can be decomposed into irreducible parts

$$|a^{(\pm)}\rangle = \sum_\alpha C_a^{(\pm)}(\alpha)\, |\alpha\rangle^{\text{irr}}\,. \tag{3.11}$$

But from this follows

$$|\Phi_a^{(\pm)}(j)\rangle = \sum_\alpha C_a^{(\pm)}(\alpha)\, |\Phi_\alpha(j)\rangle^{\text{irr}}\,. \tag{3.12}$$

Therefore if one considers for a suitable normalization the scalar product

$${}^{\text{irr}}\langle\beta\,|\,a^{(\pm)}\rangle = {}^{\text{irr}}\langle\Phi_\beta(j)|\, W\, |\Phi_a^{(\pm)}(j)\rangle = C_a^{(\pm)}(\beta) \tag{3.13}$$

one obtains an unitary mapping of the physical into functional state space. By this the S-matrix may be defined immediately

Def.:

$$S_{ab} := (\Phi_a^{(-)}(j)\,|\,\Phi_b^{(+)}(j))\,. \tag{3.14}$$

From the statements proven in the foregoing it follows immediately

Stat.: The definition of the S-matrix is invariant against the map between functional space and physical space.

Proof: The S-matrix in physical state space is defined by

$$S_{ab} = \langle a^{(-)} | b^{(+)} \rangle . \tag{3.15}$$

Substituting the decomposition of scattering states into irreducible states (3.11) one obtains

$$S_{ab} = \sum_{\alpha} C_a^{(-)}(\alpha) \, C_b^{(+)}(\alpha) . \tag{3.16}$$

The same result is obtained by substitution of (3.12) into (3.14) and by observing the unitary equivalence between the base vectors of irreducible representations. q.e.d.

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Appendix I

In this appendix we discuss the spectral representation of τ-functions. It is

$$\tau_{\substack{n\\ \alpha_1 \dots \alpha_n}}(x_1 \dots x_n) := \langle 0 | T \Psi_{\alpha_1}(x_1) \dots \Psi_{\alpha_n}(x_n) | \mu_n \rangle \tag{I.1}$$

$$= P \sum_{\lambda_1 \dots \lambda_n} \sum_{\mu_1 \dots \mu_{n-1}} (-1)^P \langle 0 | \Psi_{\alpha_{\lambda_1}}(x_{\lambda_1}) | \mu_1 \rangle \cdots \langle \mu_{n-1} | \Psi_{\alpha_{\lambda_n}}(x_{\lambda_n}) | \mu_n \rangle \, \Theta(a(x_{\lambda_1} - x_{\lambda_2})) \dots \Theta(a(x_{\lambda_{n-1}} - x_{\lambda_n}))$$

where $\Psi_\alpha(x)$ is the Hermitean field operator of the nonlinear spinor field, a a timelike four vector and $|\mu_n\rangle$ a physical state in Hilbert space. (I.1) may be written

$$\tau_{\substack{n\\ \alpha_1 \dots \alpha_n}}(x_1 \dots x_n) = P \sum_{\lambda_1 \dots \lambda_n} \sum_{\mu_1 \dots \mu_{n-1}} (-1)^P M_{\alpha_{\lambda_1} \dots \alpha_{\lambda_n}}(\mu_1 \dots \mu_n) \exp\{ i (p_0 - p_{\mu_1}) x_{\lambda_1} + \cdots$$
$$+ i (p_{\mu_{n-1}} - p_{\mu_n}) x_{\lambda_n} \} \, \Theta(a(x_{\lambda_1} - x_{\lambda_2})) \dots \Theta(a(x_{\lambda_{n-1}} - x_{\lambda_n})) \tag{I.2}$$

with

$$M_{\alpha_{\lambda_1} \dots \alpha_{\lambda_n}}(\mu_1 \dots \mu_n) := \langle 0 | \Psi_{\alpha_{\lambda_1}}(0) | \mu_1 \rangle \cdots \langle \mu_{n-1} | \Psi_{\alpha_{\lambda_n}}(0) | \mu_n \rangle . \tag{I.3}$$

Formula (I.2) may be calculated in the rest system i.e. for $a = (1, 0, 0, 0)$

$$\tau_{\substack{n\\ \alpha_1 \dots \alpha_n}}(x_1 \dots x_n) = P \sum_{\lambda_1 \dots \lambda_n} \sum_{\mu_1 \dots \mu_{n-1}} (-1)^P M_{\alpha_{\lambda_1} \dots \alpha_{\lambda_n}}(\mu_1 \dots \mu_n)$$
$$\times \exp\Big\{ -i \sum_{s=1}^{n} (\mathfrak{p}_{\mu_s} - \mathfrak{p}_{\mu_{s-1}}) \, \mathfrak{x}_{\lambda_s} - i \sum_{s=1}^{n} (p^0_{\mu_s} - p^0_{\mu_{s-1}}) t_{\lambda_s} \Big\} \, \Theta(t_{\lambda_1} - t_{\lambda_2}) \dots \Theta(t_{\lambda_{n-1}} - t_{\lambda_n}) \tag{I.4}$$

with $p := (p^0, \mathfrak{p})$. Then the Fourier transform of (I.4) reads

$$\tilde{\tau}_{\substack{n\\ \alpha_1 \dots \alpha_n}}(q_1 \dots q_n) = P \sum_{\lambda_1 \dots \lambda_n} \sum_{\mu_1 \dots \mu_{n-1}} (-1)^P M_{\alpha_{\lambda_1} \dots \alpha_{\lambda_n}}(\mu_1 \dots \mu_n)$$
$$\times \delta\Big(\sum_{s=1}^{n} \mathfrak{q}_{\lambda_s} - \mathfrak{p}_{\mu_n} \Big) \, \delta\Big(\sum_{s=1}^{n} q^0_{\lambda_s} - p^0_{\mu_n} \Big) \prod_{r=1}^{n-1} \delta\Big(\sum_{\beta=1}^{r} \mathfrak{q}_{\lambda_\beta} - \mathfrak{p}_{\mu_r} \Big) \cdot \Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta} - p^0_{\mu_r} + i \varepsilon \Big)^{-1} . \tag{I.5}$$

Proof: To prove (I.5) we apply the inverse Fourier transformation on (I.5) leading to

$$\Delta := P \sum_{\lambda_1 \dots \lambda_n} \sum_{\mu_1 \dots \mu_{n-1}} (-1)^P M_{\alpha_{\lambda_1} \dots \alpha_{\lambda_n}}(\mu_1 \dots \mu_n) \int \delta\Big(\sum_{s=1}^{n} \mathfrak{q}_{\lambda_s} - \mathfrak{p}_{\mu_n} \Big) \, \delta\Big(\sum_{s=1}^{n} q^0_{\lambda_s} - p^0_{\mu_n} \Big) \prod_{r=1}^{n-1} \delta\Big(\sum_{\beta=1}^{r} \mathfrak{q}_{\lambda_\beta} - \mathfrak{p}_{\mu_n} \Big)$$
$$\cdot \Big(\sum_{\beta=1}^{n} q^0_{\lambda_\beta} - p^0_{\mu_r} + i \varepsilon \Big)^{-1} \exp\Big\{ -i \sum_{l=1}^{n} (\mathfrak{q}_{\lambda_l} \mathfrak{x}_{\lambda_l} + q^0_{\lambda_l} t_{\lambda_l}) \Big\} \, dq_1 \dots dq_n . \tag{I.6}$$

By the substitution of the new variables

$$\sum_{\mu=1}^{r} \mathfrak{q}_{\lambda_\mu} = \mathfrak{z}_r , \qquad \mathfrak{q}_{\lambda_r} = \mathfrak{z}_r - \mathfrak{z}_{r-1} ,$$
$$\sum_{\mu=1}^{r} q^0_{\lambda_\mu} = z^0_r , \qquad q^0_{\lambda_r} = z^0_r - z^0_{r-1} \tag{I.7}$$

we obtain from (I.6)

$$\Delta = P\sum_{\lambda_1\ldots\lambda_n}\sum_{\mu_1\ldots\mu_{n-1}}(-1)^P M\underset{\alpha_{\lambda_1}\quad\alpha_{\lambda_n}}{(\mu_1\ldots\mu_n)}\int\delta(\mathfrak{z}_n-\mathfrak{p}_{\mu_n})\,\delta(z_n^0-p_{\mu_n}^0)\prod_{r=1}^{n-1}\delta(\mathfrak{z}_r-\mathfrak{p}_{\mu_r})(z_r^0-p_{\mu_r}^0+i\,\varepsilon)^{-1}$$
$$\times\exp\left\{-i\sum_{r=1}^{n}[(\mathfrak{z}_l-\mathfrak{z}_{l-1})\,\mathfrak{x}_{\lambda_l}+(z_l^0-z_{l-1}^0)\,t_{\lambda_l}]\right\}\mathrm{d}z_1\ldots\mathrm{d}z_n \tag{I.8}$$
$$= P\sum_{\lambda_1\ldots\lambda_n}\sum_{\mu_1\ldots\mu_{n-1}}(-1)^P M\underset{\alpha_{\lambda_1}\quad\alpha_{\lambda_n}}{(\mu_1\ldots\mu_n)}\int\prod_{l=1}^{n-1}\delta(\mathfrak{z}_r-\mathfrak{p}_{\mu_r})(z_r^0-p_{\mu_r}^0+i\,\varepsilon)^{-1}$$
$$\times\exp\left\{-i\sum_{l=1}^{n-1}[(\mathfrak{z}_l-\mathfrak{z}_{l-1})\,\mathfrak{x}_{\lambda_l}+(z_l^0-z_{l-1}^0)\,t_{\lambda_l}]-i[(\mathfrak{p}_{\mu_n}-\mathfrak{z}_{n-1})\,\mathfrak{x}_{\lambda_n}+(p_{\mu_n}^0-z_{n-1}^0)\,t_{\lambda_n}]\right\}\mathrm{d}z_1\ldots\mathrm{d}z_{n-1}.$$

Now we apply the auxiliary formula

$$-i\sum_{l=1}^{n-1}[(\mathfrak{z}_l-\mathfrak{z}_{l-1})\,\mathfrak{x}_{\lambda_l}+(z_l^0-z_{l-1}^0)\,t_{\lambda_l}]-i[(\mathfrak{p}_{\mu_n}-\mathfrak{z}_{n-1})\,\mathfrak{x}_{\lambda_n}+(p_{\mu_n}^0-z_{n-1}^0)\,t_{\lambda_n}]$$
$$=-i\sum_{s=1}^{n-1}[\mathfrak{z}_s(\mathfrak{x}_{\lambda_s}-\mathfrak{x}_{\lambda_{s+1}})+z_s^0(t_{\lambda_s}-t_{\lambda_{s+1}})]-i\,\mathfrak{p}_{\mu_n}\mathfrak{x}_{\lambda_n}-i\,p_{\mu_n}^0 t_{\lambda_n} \tag{I.9}$$

substitution into (I.8) and evaluation of the δ-functions then gives

$$\Delta = P\sum_{\lambda_1\ldots\lambda_n}\sum_{\mu_1\ldots\mu_{n-1}}(-1)^P M\underset{\alpha_{\lambda_1}\quad\alpha_{\lambda_n}}{(\mu_1\ldots\mu_n)}\int\prod_{r=1}^{n-1}(z_r^0-p_{\mu_r}^0+i\,\varepsilon)^{-1}$$
$$\times\exp\left\{-i\sum_{s=1}^{n-1}[\mathfrak{p}_{\mu_s}(\mathfrak{x}_{\lambda_s}-\mathfrak{x}_{\lambda_{s+1}})+z_s^0(t_{\lambda_s}-t_{\lambda_{s+1}})]-i\,\mathfrak{p}_{\mu_n}\mathfrak{x}_{\lambda_n}-i\,p_{\mu_n}^0 t_{\lambda_n}\right\}\mathrm{d}z_1^0\ldots\mathrm{d}z_{n-1}^0. \tag{I.10}$$

In this integral poles occur for $z_r^0=p_{\mu_r}^0-i\,\varepsilon$ in the complex z_r^0-plane. Applying the residual integration the contour has to be closed in the upper half plane for $t_{\lambda_s}-t_{\lambda_{s+1}}<0$ resulting in $\Delta=0$. For $t_{\lambda_s}-t_{\lambda_{s+1}}>0$ the contour has to be closed in the lower half plane giving $\Delta=\Delta_0\neq 0$. Therefore for arbitrary $t_{\lambda_s}-t_{\lambda_{s+1}}$ the result may be written

$$\Delta=\Delta_0\,\Theta(t_{\lambda_1}-t_{\lambda_2})\ldots\Theta(t_{\lambda_{n-1}}-t_{\lambda_n}), \tag{I.11}$$

with

$$\Delta_0 := P\sum_{\lambda_1\ldots\lambda_n}\sum_{\mu_1\ldots\mu_{n-1}}(-1)^P M\underset{\alpha_{\lambda_1}\quad\alpha_{\lambda_n}}{(\mu_1\ldots\mu_n)}$$
$$\times\exp\left\{-i\sum_{s=1}^{n-1}[\mathfrak{p}_{\mu_s}(\mathfrak{x}_{\lambda_s}-\mathfrak{x}_{\lambda_{s+1}})+p_{\mu_s}^0(t_{\lambda_s}-t_{\lambda_{s+1}})]-i\,\mathfrak{p}_{\mu_n}\mathfrak{x}_{\lambda_n}-i\,\mathfrak{p}_{\mu_n}^0 t_{\lambda_n}\right\}. \tag{I.12}$$

Applying now the auxiliary formula

$$-i\sum_{s=1}^{n-1}p_{\mu_s}(x_{\lambda_s}-x_{\lambda_{s+1}})-i\,p_{\mu_n}x_{\lambda_n}=i\sum_{s=1}^{n}(p_{\mu_{s-1}}-p_{\mu_s})\,x_{\lambda_s}. \tag{I.13}$$

(I.11), (I.12) is transformed into

$$\Delta = P\sum_{\lambda_1\ldots\lambda_n}\sum_{\mu_1\ldots\mu_{n-1}}(-1)^P M\underset{\alpha_{\lambda_1}\quad\alpha_{\lambda_n}}{(\mu_1\ldots\mu_n)}\exp\left\{i\sum_{s=1}^{n}(p_{\mu_{s-1}}-p_{\mu_s})\,x_{\lambda_s}\right\}\Theta(t_{\lambda_1}-t_{\lambda_2})\ldots\Theta(t_{\lambda_{n-1}}-t_{\lambda_n}) \tag{I.13}$$
$$=\tau_n\underset{\alpha_1\quad\alpha_n}{(x_1\ldots x_n)} \qquad \text{q.e.d.}$$

The Fourier transformed τ_n will be used to discuss the scalar products of state functionals.

Appendix II

To evaluate the functional scalar product of Section 3 we do not discuss the normal functionals $|\Phi(j)\rangle$, but the timeordered functionals

$$|\mathfrak{T}(j)\rangle := e^{-jFj}\,|\Phi(j)\rangle \tag{II.1}$$

where F is the vacuum two point function of the theory. Because of the simple connection between normal- and timeordered functionals any statement about the timeordered functionals can be applied equally well to normal functionals. It is

$$|\mathfrak{T}(j)\rangle = \sum_{n=0}^{\infty} \tau_n(\underset{\alpha_1}{x_1}\dots\underset{\alpha_n}{x_n})\,|D_n(\overset{\alpha_1}{x_1}\dots\overset{\alpha_n}{x_n})\rangle\,. \tag{II.2}$$

Observing the definition of W and the scalar product of power functionals

$$\langle D_k(\underset{\varkappa_1}{u_1}\dots\underset{\varkappa_k}{u_k})|\,D_m(\overset{\beta_1}{y_1}\dots\overset{\beta_m}{y_m})\rangle = \frac{1}{(m!)^2}\,\delta_{km}\,P\sum_{\lambda_1\dots\lambda_m}(-1)^P\,\delta^{\beta_{\lambda_1}}_{\varkappa_1}\,\delta(u_1-y_{\lambda_1})\dots\delta^{\beta_{\lambda_m}}_{\varkappa_m}\,\delta(u_m-y_{\lambda_m}) \tag{II.3}$$

we obtain

$$\langle\mathfrak{T}_a(j)|\,W\,|\mathfrak{T}_b(j)\rangle = \sum_n \varepsilon^n\,\frac{1}{(n!)^2}\,\overset{\times}{\tau}_n(\overset{\alpha_1}{x_1}\dots\overset{\alpha_n}{x_n})\,\tau_n(\underset{\alpha_1}{x_1}\dots\underset{\alpha_n}{x_n}) = \sum_n \frac{\varepsilon^n}{(n!)^2}\,\tilde{\overset{\times}{\tau}}_n(\overset{\alpha_1}{q_1}\dots\overset{\alpha_n}{q_n})\,\tilde{\tau}_n(\underset{\alpha_1}{q_1}\dots\underset{\alpha_n}{q_n})\,. \tag{II.4}$$

Now the statement (3.3) reads

$$\lim_{\varepsilon\to 0}\langle\mathfrak{T}_a(j)|\,W\,|\mathfrak{T}_b(j)\rangle = \sum_\nu A_\nu(a,b)\,\delta^\nu(0)\,. \tag{II.5}$$

Proof: By (II.4) we evaluate (II.5) in Fourier space by using (I.5). To prove (II.5) it is sufficient to show, that (II.5) contains at least $A_0(a,b)\neq 0$ and at least one $A_\nu(a,b)\neq 0$ with $\nu\neq 0$. We obtain

$$\begin{aligned}
&\langle\mathfrak{T}_a(j)|\,W\,|\mathfrak{T}_b(j)\rangle\\
&\quad= \sum_n \varepsilon^n\,\frac{1}{(n!)^2}\sum_{\substack{\lambda_1\dots\lambda_n\\ \lambda_1'\dots\lambda_n'}}\ \sum_{\substack{\mu_1\dots\mu_{n-1}\\ \mu_1'\dots\mu'_{n-1}}} M(\overset{\alpha_{\lambda 1}}{\mu_1}\dots\overset{\alpha_{\lambda n}}{\mu_n})\,M(\underset{\alpha_{\lambda 1'}\dots\alpha_{\lambda n'}}{\mu_1'\dots\mu_n'})\int \mathrm{d}\mathfrak{q}_1\dots\mathrm{d}\mathfrak{q}_n\,\mathrm{d}q_1^0\dots\mathrm{d}q_n^0\,\delta\Big(\sum_{s=1}^n \mathfrak{q}_{\lambda_s}-\mathfrak{p}_{\mu_n}\Big)\\
&\qquad\times\delta\Big(\sum_{s=1}^n q^0_{\lambda_s}-p^0_{\mu_n}\Big)\prod_{r=1}^{n-1}\delta\Big(\sum_{\beta=1}^{n-1}\mathfrak{q}_{\lambda_\beta}-\mathfrak{p}_{\mu_r}\Big)\Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta}-p^0_{\mu_r}-i\,\varepsilon\Big)^{-1}\delta\Big(\sum_{s=1}^{n-1}\mathfrak{q}_{\lambda_s'}-\mathfrak{p}_{\mu_n'}\Big)\,\delta\Big(\sum_{s=1}^{n} q^0_{\lambda_s'}-p^0_{\mu_n'}\Big)\\
&\qquad\qquad\times\prod_{r=1}^{n-1}\delta\Big(\sum_{\beta=1}^{r}\mathfrak{q}_{\lambda_\beta'}-\mathfrak{p}_{\mu_r'}\Big)\Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta'}-p^0_{\mu_r'}+i\,\varepsilon\Big)^{-1}
\end{aligned} \tag{II.6}$$

$$\begin{aligned}
&= \sum_n \varepsilon^n\,\frac{1}{(n!)^2}\sum_{\substack{\lambda_1\dots\lambda_n\\ \lambda_1'\dots\lambda_n'}}\ \sum_{\substack{\mu_1\dots\mu_{n-1}\\ \mu_1'\dots\mu'_{n-1}}} M(\overset{\alpha\lambda_1}{\mu_1}\dots\overset{\alpha\lambda_n}{\mu_n})\,M(\underset{\alpha\lambda_1'}{\mu_1'}\dots\underset{\alpha\lambda_n'}{\mu_n'})\,\delta(p_{\mu_n}-p_{\mu_n}')\int\mathrm{d}\mathfrak{q}_1\dots\mathrm{d}\mathfrak{q}_n\,\mathrm{d}q_1^0\dots\mathrm{d}q_n^0
\end{aligned} \tag{II.7}$$

$$\begin{aligned}
&\times\delta\Big(\sum_{s=1}^n q_{\lambda_s}-p_{\mu_n}\Big)\prod_{r=1}^{n-1}\delta\Big(\sum_{\beta=1}^{r}\mathfrak{q}_{\lambda_\beta}-\mathfrak{p}_{\mu_r}\Big)\,\delta\Big(\sum_{\beta=1}^{r}\mathfrak{q}_{\lambda_\beta'}-\mathfrak{p}_{\mu_r'}\Big)\Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta}-p^0_{\mu_r}-i\,\varepsilon\Big)^{-1}\Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta'}-p^0_{\mu_r'}+i\,\varepsilon\Big)^{-1}\\
&= \sum_n \varepsilon^n\,\frac{1}{(n!)^2}\sum_{\substack{\lambda_1\dots\lambda_n\\ \lambda_1'\dots\lambda_n'}}\ \sum_{\substack{\mu_1\dots\mu_{n-1}\\ \mu_1'\dots\mu'_{n-1}}} M(\overset{\alpha\lambda_1}{\mu_1}\dots\overset{\alpha\lambda_n}{\mu_n})\,M(\underset{\alpha\lambda_1'}{\mu_1'}\dots\underset{\alpha\lambda_n'}{\mu_n'})\,\delta(p_{\mu_n}-p_{\mu_n'})\int\mathrm{d}q_1\dots\mathrm{d}q_{n-1}\\
&\quad\times\prod_{r=1}^{n-1}\delta\Big(\sum_{\beta=1}^{r}\mathfrak{q}_{\lambda_\beta}-\mathfrak{p}_{\mu_r}\Big)\,\delta\Big(\sum_{\beta=1}^{r}\mathfrak{q}_{\lambda_\beta'}-\mathfrak{p}_{\mu_r'}\Big)\Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta}-p^0_{\mu_r}-i\,\varepsilon\Big)^{-1}\Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta'}-p^0_{\mu_r'}+i\,\varepsilon\Big)^{-1}.
\end{aligned} \tag{II.8}$$

The permutations $\lambda_1\dots\lambda_n$ can be eliminated. From (II.8) then follows

$$\begin{aligned}
\langle\mathfrak{T}_a|\,W\,|\mathfrak{T}_b\rangle &= \sum_n \varepsilon^n\,\frac{1}{n!}\sum_{\lambda_1'\dots\lambda_n'}\ \sum_{\substack{\mu_1\dots\mu_{n-1}\\ \mu_1'\dots\mu'_{n-1}}} M(\overset{\alpha_1}{\mu_1}\dots\overset{\alpha_n}{\mu_n})\,M(\underset{\alpha\lambda_1'}{\mu_1'}\dots\underset{\alpha\lambda_n'}{\mu_n'})\,\delta(p_{\mu_n}-p_{\mu_n'})\\
&\quad\times\int\mathrm{d}q_1\dots\mathrm{d}q_{n-1}\prod_{r=1}^{n-1}\delta\Big(\sum_{\beta=1}^{r}\mathfrak{q}_\beta-\mathfrak{p}_{\mu_r}\Big)\,\delta\Big(\sum_{\beta=1}^{r}\mathfrak{q}_{\lambda_\beta'}-\mathfrak{p}_{\mu_r'}\Big)\Big(\sum_{\beta=1}^{r} q^0_\beta-p^0_{\mu_r}-i\,\varepsilon\Big)^{-1}\Big(\sum_{\beta=1}^{r} q^0_{\lambda_\beta'}-p^0_{\mu_r'}+i\,\varepsilon\Big)
\end{aligned} \tag{II.9}$$

Further the $q_1^0 \ldots q_{n-1}^0$ integration paths may be completed to closed contours in the corresponding complex planes. This gives

$$\int \prod_{r=1}^{n-1} \left(\sum_{\beta=1}^{r} q_\beta^0 - p_{\mu_r}^0 - i\,\varepsilon \right)^{-1} \left(\sum_{\beta=1}^{r} q_{\lambda_{\beta'}}^0 - p_{\mu_{r'}}^0 + i\,\varepsilon \right)^{-1} \mathrm{d}q_1^0 \ldots \mathrm{d}q_{n-1}^0$$
$$= \frac{1}{(2\,\pi\, i)^{n-1}} \oint \prod_{r=1}^{n-1} \left(\sum_{\beta=1}^{r} q_\beta^0 - p_{\mu_r}^0 - i\,\varepsilon \right)^{-1} \left(\sum_{\beta=1}^{r} q_{\lambda_{\beta'}}^0 - p_{\mu_{r'}}^0 + i\,\varepsilon \right)^{-1} \mathrm{d}q_1^0 \ldots \mathrm{d}q_{n-1}^0 . \quad \text{(II.10)}$$

This poles in the upper half plane are $q_s^0 = \sum_{k=1}^{s} (-1)^{s+k} (p_{\mu_k}^0 + i\,\varepsilon)$. Therefore one obtains

$$\int \prod_{r=1}^{n-1} \left(\sum_{\beta=1}^{r} q_\beta^0 - p_{\mu_r}^0 - i\,\varepsilon \right)^{-1} \left(\sum_{\beta=1}^{r} q_{\lambda_{\beta'}}^0 - p_{\mu_{r'}}^0 + i\,\varepsilon \right)^{-1} \mathrm{d}q_1^0 \ldots \mathrm{d}q_{n-1}^0$$
$$= \frac{1}{(2\,\pi\, i)^{n-1}} \prod_{r=1}^{n-1} \left(\sum_{\beta=1}^{r} \sum_{k=1}^{\lambda_{\beta'}} (-1)^{\lambda_{\beta'}+k} (p_{\mu_k}^0 + i\,\varepsilon) - p_{\mu_{r'}}^0 + i\,\varepsilon \right)^{-1} \quad \text{(II.11)}$$

By integration of the δ-functions follows $\mathfrak{q}_s = \sum_{k=1}^{s} (-1)^{s+k}\, \mathfrak{p}_{\mu_k}$ and therefore

$$\langle \mathfrak{T}_a | W | \mathfrak{T}_b \rangle = \sum_n \varepsilon^n \frac{1}{n!} \sum_{\lambda_1' \ldots \lambda_n'} \sum_{\substack{\mu_1 \ldots \mu_{n-1} \\ \mu_1' \ldots \mu'_{n-1}}} M(\overset{\alpha_1}{\mu_1} \ldots \overset{\alpha_n}{\mu_n})\, M(\underset{\alpha\lambda_1'}{\mu_1'} \ldots \underset{\alpha\lambda_n'}{\mu_n'})\, \delta(p_{\mu_n} - p_{\mu_n'})$$
$$\times \prod_{r=1}^{n-1} \delta\left(\sum_{\beta=1}^{r} \sum_{k=1}^{\lambda_{\beta'}} (-1)^{\lambda_{\beta'}+k}\, \mathfrak{p}_{\mu_k} - \mathfrak{p}_{\mu_{r'}} \right) \left(\sum_{\beta=1}^{r} \sum_{k=1}^{\lambda_{\beta'}} (-1)^{\lambda_{\beta'}+k}\, p_{\mu_k}^0 - p_{\mu_{r'}}^0 + i\,\varepsilon \right)^{-1} . \quad \text{(II.12)}$$

Due to the δ-functions and the limes procedure $\varepsilon \to 0$ only four momentum vectors are admitted satisfying the conditions

$$\sum_{\beta=1}^{r} \sum_{k=1}^{\lambda_{\beta'}} (-1)^{\lambda_{\beta'}+k}\, p_{\mu_k} = p_{\mu_{r'}}, \qquad r = 1, \ldots, n-1 . \quad \text{(II.13)}$$

This is equivalent with

$$\sum_{k=1}^{\lambda_{r'}} (-1)^{\lambda_{r'}-k}\, p_{\mu_k} = \sum_{k=1}^{r} (-1)^{r-k}\, p_{\mu_{k'}}, \qquad r = 1, \ldots, n-1 . \quad \text{(II.14)}$$

By this (II.11) goes over into

$$\langle \mathfrak{T}_a | W | \mathfrak{T}_b \rangle = \sum_n \varepsilon^n \frac{1}{n!} \sum_{\lambda_1' \ldots \lambda_r'} \sum_{\substack{\mu_1 \ldots \mu_{n-1} \\ \mu_1' \ldots \mu'_{n-1}}} M(\overset{\alpha_1}{p_{\mu_1}} \ldots p_{\mu_{n-1}} \overset{\alpha_n}{p_{\mu_n}})\, M(\underset{\alpha\mu_1'}{p_{u_1'}} \ldots p_{u'_{n-1}} \underset{\alpha\lambda_n'}{p_{u'_n}})\, \delta(p_{\mu_n} - p_{\mu_n'})$$
$$\prod_{r=1}^{n-1} \delta\left(\sum_{k=1}^{\lambda_{r'}} (-1)^{\lambda_{r'}-k}\, \mathfrak{p}_{\mu_k} - \sum_{k=1}^{r} (-1)^{r-k}\, \mathfrak{p}_{\mu'_k} \right) \left(\sum_{k=1}^{\lambda_{r'}} (-1)^{\lambda_{r'}-k}\, p_{\mu_k}^0 - \sum_{k=1}^{r} (-1)^{r-k}\, p_{\mu_k'}^0 + i\,\varepsilon \right)^{-1} . \quad \text{(II.15)}$$

In (II.15) the summations over the intermediate quantumstates μ_α, μ_α' have been replaced by summations over four momentums, due to the symmetry properties of the theory. No other quantum numbers are considered additionally, as they are of no importance for our proof. Applying the substitution

$$\sum_{k=1}^{r} (-1)^{r-k}\, p_{\mu_k} = z_{\mu_r}; \qquad p_{\mu_r} = z_{\mu_r} + z_{\mu_{r-1}} . \quad \text{(II.16)}$$

(II.15) goes over into

$$\langle \mathfrak{T}_a | W | \mathfrak{T}_b \rangle = \sum_n \varepsilon^n \frac{1}{n!} \sum_{\lambda_1' \ldots \lambda_n'} \sum_{\substack{\mu_1 \ldots \mu_{n-1} \\ \mu_1' \ldots \mu'_{n-1}}} \tilde{M}(\overset{\alpha_1}{z_{\mu_1}} \ldots z_{\mu_{n-1}} \overset{\alpha_n}{p_{\mu_n}})\, \tilde{M}(\underset{\alpha\mu_1'}{z_1'} \ldots z_{\mu'_{n-1}} \underset{\alpha\lambda_n'}{p_{\mu_n'}^0})$$
$$\times\, \delta(p_{\mu_n} - p_{\mu_n'}) \prod_{r=1}^{n-1} \delta(\mathfrak{z}_{\mu\lambda_{r'}} - \mathfrak{z}_{\mu_{r'}})\, (z_{\mu\lambda_{r'}}^0 - z_{\mu_{r'}}^0 + i\,\varepsilon)^{-1} . \quad \text{(II.17)}$$

The summation can be subdivided into a part $\sum^1$ containing no vacuum state and the rest part $\sum^2$ where the intermediate states contain the vacuum state $|0\rangle$ of the theory

$$\sum_{\substack{\mu_1\ldots\mu_{n-1}\\ \mu_1'\ldots\mu'_{n-1}}} = \sum^1_{\substack{\mu_1\ldots\mu_{n-1}\\ \mu_1'\ldots\mu'_{n-1}}} + \sum^2_{\substack{\mu_1\ldots\mu_{n-1}\\ \mu_1'\ldots\mu'_{n-1}}} . \tag{II.18}$$

Due to symmetry properties $\sum^1$ contains for any intermediate state an integration over the momentum space. Therefore (II.17) can be written

$$\langle \mathfrak{T}_a | W | \mathfrak{T}_b \rangle = \sum_n \frac{1}{n!} \sum^1_{\substack{\lambda_1'\ldots\lambda_n'\\ \mu_1\ldots\mu_{n-1}}} \tilde{M}^{\alpha_1 \ldots \alpha_{n-1}\,\alpha_n}(z_{\mu_1}\ldots z_{\mu_{n-1}}\, p_{\mu_n})\, \tilde{M}_{\alpha\lambda_1' \ldots \alpha\lambda'_{n-1}\,\alpha\lambda_n'}(z_{\mu\lambda_1'}\ldots z_{\mu\lambda'_{n-1}}\, p_{\mu_n'})\, \delta(p_{\mu_n} - p_{\mu_n'}) \tag{II.19}$$

$$+ \sum_n \varepsilon^n \frac{1}{n!} \sum_{\lambda_1'\ldots\lambda_n'} \sum_{\substack{\mu_1\ldots\mu_{n-1}\\ \mu_1'\ldots\mu'_{n-1}}} \tilde{M}^{\alpha_1\ldots\alpha_n}(z_{\mu_1}\ldots p_{\mu_n})\, \tilde{M}_{\alpha\lambda_1'\ldots\alpha\lambda_n'}(z_{\mu_1'}\ldots p_{\mu_n'})\, \delta(p_{\mu_n} - p_{\mu_n'}) \prod_{r=1}^{n-1} \delta(\mathfrak{z}_{\mu\lambda_r'} - \mathfrak{z}_{\mu_r'})(z^0_{\mu\lambda_r'} - z^0_{\mu_r'} + i\,\varepsilon)^{-1}$$

From the first term in (II.19) all δ-functions are eliminated and only the momentum conservation is present. This term therefore contributes partly to $A_0(a, b)$. Now we show, that $\sum^2$ contains at least one $\delta^2(0)$ term. By construction $\sum^2$ contains the vacuum state. We assume $|p_{\mu_\varkappa'}\rangle \equiv |0\rangle$. Then from (II.16) follows $z_{\mu_\varkappa'} = -z_{\mu'_{\varkappa-1}}$ and one obtains

$$\sum_n \varepsilon^n \frac{1}{n!} \sum_{\lambda_1'\ldots\lambda_n'} \sum^2 (\quad) = \sum_n \varepsilon^n \frac{1}{n!} \sum_{\lambda_1'\ldots\lambda_n'} \sum^2_{\substack{\mu_1\ldots\mu_{n-1}\\ \mu_1'\ldots\mu'_{n-1}}} \tilde{M}^{\alpha_1\ldots\alpha_n}(z_{\mu_1}\ldots z_{\mu_{n-1}}\, p_{\mu_n})\, \tilde{M}_{\alpha\lambda_1'\ldots\alpha\lambda_n'}(z_{\mu_1}\ldots z_{\mu'_{n-1}}\, p_{\mu_n})\, \delta(p_{\mu_n} - p_{\mu_n'})$$

$$\times\, \delta(\mathfrak{z}_{\mu\lambda_\varkappa} - \mathfrak{z}_{\mu_\varkappa'})\, \delta(\mathfrak{z}_{\mu\lambda'_{\varkappa-1}} + \mathfrak{z}_{\mu_\varkappa'}) \prod_{\substack{r=1\\ \neq\varkappa,\varkappa-1}}^{n-1} \delta(\mathfrak{z}_{\mu\lambda_r'} - \mathfrak{z}_{\mu_r'})(z^0_{\mu\lambda_r'} - z^0_{\mu_r'} + i\,\varepsilon)^{-1}. \tag{II.20}$$

Among all permutations λ_ϱ' the values $\lambda_\varkappa' = \varkappa$, $\lambda_{\varkappa-1}' = \varkappa - 1$ have to occur. Also for the summations terms with $\mu_\varkappa = \mu_\varkappa'$ and $\mu_{\varkappa-1} = \mu_{\varkappa-1}'$ have to occur. But then follows

$$\delta(\mathfrak{z}_{\mu\lambda\varkappa'} - \mathfrak{z}_{\mu_\varkappa'})\, \delta(\mathfrak{z}_{\mu\lambda'_{\varkappa-1}} + \mathfrak{z}_{\mu_\varkappa'}) = \delta(\mathfrak{z}_{\mu\lambda_\varkappa'} + \mathfrak{z}_{\mu\lambda'_{\varkappa-1}})\, \delta(\mathfrak{z}_{\mu\lambda'_{\varkappa-1}} + \mathfrak{z}_{\mu_\varkappa'})$$
$$= \delta(\mathfrak{z}_{\mu_\varkappa'} + \mathfrak{z}_{\mu'_{\varkappa-1}})\, \delta(\mathfrak{z}_{\mu'_{\varkappa-1}} + \mathfrak{z}_{\mu_\varkappa'}) = \delta^2(0)\,. \tag{II.21}$$

By this it is shown, that $\sum^2$ contains at least one $\delta^2(0)$ term. (II.5) therefore has to be a power series in $\delta(0)$ q.e.d.